LSU Dept. of Physics and Astronomy
Qualifying Exam
Quantum Mechanics Question Bank
(09/2023)

1. Indicate whether the following statements are true or false. Do not give your response just by guessing because a correct answer will draw 1 point, an incorrect one -1 point and no answer will count for zero points.

   i) The WKB method was devised for rapidly varying potentials.

   ii) The energy of a one-dimensional harmonic oscillator perturbed by a potential linear in x can be calculated exactly.

   iii) The spin-orbit interaction in atoms is relativistic in origin.

   iv) If the spin-orbit interaction is neglected, the z component of the angular momenta of individual electrons in an atom would each be a good quantum number but the z component of the total angular momentum of all the electrons would not.

   v) The separation between the ${}^2P_{1/2}$ and ${}^2P_{3/2}$ levels in the hydrogen atom is due to radiative effects and is called the Lamb shift.

   vi) s-wave scattering is most significant for large kinetic energies of the incident particles.

   vii) The total scattering cross-section for elastic scattering by a short-range potential is all that one needs to know to obtain all the particle wave phase shifts.

   viii) In a self-consistent one electron potential for an atom, the exchange term is attractive while the direct term is repulsive.

   ix) The Bohr orbit for a μ^- bound to a proton is very much smaller than that of an electron bound to a proton.

   x) The result that most atoms do not exhibit a first order Stark affect has to do with the law of conservation of parity.

2. Using the first Born Approximation, find the differential scattering cross section for the exponential potential $V = - V_o e^{-r/a}$. Sketch the angular dependence of the scattering amplitude.

   Hint: $\int_o^\infty r \sin \rho r e^{-\alpha r} dr = \dfrac{2\alpha\rho}{\left(\alpha^2 + \rho^2\right)^2}$

3. All parts of this question call for quick and short answers.

(a) [2 points] Which of the following are eigenstates of parity. For those that are, what is the eigenvalue?
$e^{-k|x|}$, $x^6 + 3x^2 + 1$, $\sin 4x + 2 \sin 3x$, $\cos(3x + 7)$, $H_3(x)$, the last being a Hermite Polynomial.

(b) [3 points] Identify the Hermitian operators in the following
$i\partial^3/\partial x^3$, $p^2 + x^3$, xp^2, xpx^2px, $[p^2, x^2]$

(c) [3 points] Given that the peak wavelength of a 6000 K blackbody (surface of Sun) lies at 500 nm, what is the peak wavelength of the 3 K cosmic background radiation? Express also as a frequency.

(d) [3 points] A crystal has planar spacing of 0.3 nm. What order of magnitude of kinetic energies would you design for an electron microscope for obtaining good diffraction images?

(e) [3 points] The discovery of the ψ particle at 3.1 GeV/c^2 as a resonance of very narrow width of 50 keV came as a surprise. What is the implied lifetime of the particle that caused this surprise?

(f) [3 points] Evaluate $[x^2, [x^2, p^2]]$

(g) [3 points] For a one-dimensional harmonic oscillator of mass m and frequency ω, evaluate the matrix element among stationary states: $\langle n + 2|xp|n\rangle$.

4. Let $|f_1\rangle$, $|f_2\rangle$, and $|f_3\rangle$ be defined by the equations below:

$$S^2|f_1\rangle = 2\hbar^2|f_1\rangle \qquad S_z|f_1\rangle = +1\hbar|f_1\rangle,$$

$$S^2|f_2\rangle = 2\hbar^2|f_2\rangle \qquad S_z|f_2\rangle = +0\hbar|f_2\rangle,$$

$$S^2|f_3\rangle = 2\hbar^2|f_3\rangle \qquad S_z|f_3\rangle = -1\hbar|f_3\rangle,$$

Consider an ensemble of six quantum mechanical systems; one of which is in the state $|f_1\rangle$, two of which are in the state $|f_2\rangle$, and three of which are in the state $|f_3\rangle$.

a) Construct explicitly the density matrix for the ensemble.

b) By direct manipulation of the density matrix and other quantum mechanical operators, calculate the ensemble average for S_z.

5. A free particle with energy "E" and spin 1/2 is traveling in the x-direction. The spin of the particle also points in the x-direction. Beginning at x = 0 there is a spin dependent potential of the form

$$V(x) = V_0(1 + \alpha\sigma_z),$$

where V_0 and α are constants, $|\alpha| < 1$, and σ_z is the usual Pauli matrix.

a) Assuming that the energy "E" is smaller than either value of V(x), calculate an expression for the reflected wave.

b) Prove that the spin of the reflected wave lies entirely in the x-y plane.

c) Calculate expectation values for σ_z and σ_y for the reflected wave. By visual inspection of your answers determine when the expectation value for σ_y will be zero. Under these conditions what will be the expectation value for σ_x? Is your answer physically plausible?

6 A charged, linear harmonic oscillator is created in its ground state at time $t = -\infty$. Immediately thereafter a weak but time dependent electric field is turned on. The field is:

$$\vec{E}(t) = \hat{i}\left(\frac{N}{\tau}\right)\frac{1}{\sqrt{\pi}}\exp\left(-t^2/\tau^2\right) \quad ,$$

where N and τ are constants.

a) What transitions will be induced by this perturbation?

b) What is the probability that the oscillator will be found in the first excited state at $t = +\infty$?

c) Examine your answer in the limits $\tau \to 0$ and $\tau \to \infty$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\int_0^{+\infty} dx e^{-ax^2}\cos(bx) = \frac{1}{2}\left(\sqrt{\frac{\pi}{a}}\right)\exp(-b^2/4a)$$

7. a) Write down the 3×3 matrices that represent the operators L_x, L_y and L_z of angular momentum for a value of $\ell = 1$ in a basis which has L_z diagonal.

b) An arbitrary rotation of the state of such a system can be described by the operator

$$U = \exp(-i\alpha L_z/\hbar)\exp(-i\beta L_y/\hbar)\exp(-i\gamma L_z/\hbar) \, ,$$

where (α, β, γ) are the Euler angles describing the rotation. Using a), construct a 3×3 matrix representation of U.

c) Find the expectation value, $\langle \vec{L} \rangle$, in the state which results from applying U to an initial $\ell = 1,\ m = 1$ state.

8. All parts of this question call for quick and short answers.

(a) [2 points] Which of the following are eigenstates of parity. For those that are, what is the eigenvalue?

$$e^{-r^2}\sin\theta\cos\phi,\ Y_0^0+Y_3^1,\ z(3x^2-y^4),\ r+4yz-r\sin\theta$$

(b) [2 points] Give the column vector representation of

$$\psi = \frac{1}{\sqrt{10}}(Y_2^2 - 2Y_2^1 + 2Y_2^{-1} + Y_2^{-2}).$$

(c) [2 points] For two non-identical d electrons, what are the possible ${}^{2S+1}L_J$ states?

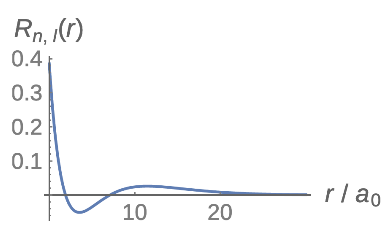
(d) [2 points] Which of the above states are allowed if they were identical d electrons?

(e) [2 points] Between which of the following pairs of hydrogen nlm states is an electric dipole transition allowed? $3p1$ and $2p0$, $3d1$ and $5p1$, $3d0$ and $2s0$, $3p1$ and $2s0$.

(f) [2 points] What is the wavelength of the $3p \to 2s$ Balmer-α line in hydrogen?

(g) [2 points] The bottom of a square well is perturbed as shown. Identify for each case, the first non-trivial order of perturbation of energy levels, and also its sign for the ground state.

(h) [2 points] Identify the n and l quantum numbers for the radial wave function of the hydrogen atom:



(i) [2 points] Positronium is a bound state of an electron and positron analogous to the hydrogen atom. What are its radius and ground state energy?

(j) [2 points] Is e^{-ax} a suitable trial wave function for variationally estimating the ground state energy of an attractive one-dimensional potential? Give reasons.

9. Consider a particle confined in a two dimensional box

V(x,y) = 0   0<x<L, and   0<y<L

V(x,y) = ∞   otherwise

The eigenstates and eigenenergies are given by

$$\varphi_{np}(x,y) = \tfrac{2}{L}\sin\left(\tfrac{n\pi x}{L}\right)\sin\left(\tfrac{p\pi y}{L}\right)$$
$$E_{np} = E_1(n^2 + p^2)$$

Consider a perturbation

H'(x,y) = V_0 for $\frac{L}{2} - \frac{a}{2} < y < \frac{L}{2} + \frac{a}{2}$

a) Find the first order correction to the energy of the ground state.

b) Find the first order correction to the eigenfunction of the ground state.

c) The first excited energy level has a two fold degeneracy. Find the splitting of the energy level due to the perturbation in first order.

Note: Use the approximation a<<L when evaluating integrals

10. Consider two identical spin-1/2 particles of mass m in a one-dimensional box where

$$V(x) = \begin{cases} 0 \text{ for } 0 < x < L \\ \infty \quad x < 0 \\ \infty \quad x > L \end{cases}$$

The possible energy levels are

$$E = \left(n_1^2 + n_2^2\right)E_1 \text{ where } E_1 = \frac{\hbar^2\pi^2}{2mL^2} \quad .$$

Write down the properly symmetrized and normalized eigenfunctions for

a) $E = 2E_1$

b) $E = 5E_1$.

11. A particle of mass m and energy $E > 0$ is incident from the left on a potential barrier given by

$$V(x) = 0 \qquad x < 0$$
$$V(x) = V_o(1 - x/a) \quad x \geq 0$$

where V_o is an energy and a is a length.
In the limit that $E << V_o$, determine the energy dependence of the transmission probability.

Hint: You may use the semiclassical approximation in this limit. To get the energy dependence you do not have to evaluate any integrals.

12. In a certain triangular molecule, an electron is free to hop from site to site. The resulting eigentstates and eigenergies for the electron are:

$$E_1 = -2e \qquad |\psi_1\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right)$$

$$E_2 = +e \qquad |\psi_2\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + e^{i2\pi/3}|2\rangle + e^{i4\pi/3}|3\rangle\right)$$

$$E_3 = +e \qquad |\psi_3\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + e^{-i2\pi/3}|2\rangle + e^{-i4\pi/3}|3\rangle\right)$$

a) At $t = 0$, the state vector is

$$|\psi(0)\rangle = \frac{1}{\sqrt{6}}(2|1\rangle + |2\rangle + |3\rangle).$$

The energy is measured. What values can be found and with what probabilities?

b) Determine $|\psi(t)\rangle$ given $|\psi(0)\rangle$ above.

13. All parts of this question call for quick, short answers.

a) Estimate the lowest energy possible when an electron is confined to a distance of 10^{-8} cm.

b) The 589 nm yellow line of sodium arises from an excited state with a lifetime for optical emission of 10^{-8} seconds. Estimate the natural width of the line.

c) Evaluate $< jm|J_x^2|\, jm >$.

d) Find the eigenvalues of $\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$.

Hint: Think of the matrix as an operator in five-dimensional space, and think in terms of subspaces.

e) What are the possible values of the total angular momentum j for a d-electron? In each these states, evaluate $\vec{\ell}\cdot\vec{s}\,|\,j>$.

14. A particle is moving in the x-y plane subject to a uniform magnetic field in the x-direction, expressed by a vector potential $\vec{A}=(0,Bx,0)$.

a) What is the Hamiltonian of this system?

b) Show that the operator $\hat{p}_y$ is a constant of the motion.

c) Find the allowed energies and eigenfunctions.

d) Explain why the energies are degenerate.

15. Two identical spin –1/2 particles of mass m moving in one dimension have the Hamiltonian.

$$H=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{\lambda}{m}\delta(\vec{r}_1-\vec{r}_2)\vec{s}_1\cdot\vec{s}_2$$

where $(p_i,\ \vec{r}_i,\ \vec{s}_i)$ are the momentum, position and spin operators for the i-th particle.

a) What operators, besides the Hamiltonian, are constants of the motion and provide good quantum numbers for the stationary states?

b) What are the symmetry requirements for the spin and spatial wave functions?

c) If $\lambda > 0$, find the energy and quantum numbers for the bound state.

16. A particle of mass m moves in a one-dimensional potential of the form

$$V(x)=\begin{cases} Fx & x\ge 0 \\ \infty & x<0 \end{cases}$$

a) Write down the Schrödinger equation for this problem and state the boundary conditions on the wave function.

b) Estimate the ground state energy using a variational wave function ψ(x) = x exp(-ax).

c) Estimate the ground state energy using Bohr-Sommerfeld/WKB quantization. Pay attention to the behavior near each turning point.

17. Two particles of masses m_1 and m_2 are restricted to move in one-dimension and have coordinates and momenta $x_i, p_i, i = 1,2$. The Hamiltonian of the system is given by

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2} m_1 w^2 x_1^2 + \frac{1}{2} m_2 w^2 x_2^2 + \frac{1}{2} k(x_1 - x_2)^2 ,$$

where $k > 0$, and k and w are constants.

a) In a couple of lines, state what, physically, this Hamiltonian describes.

b) Obtain the energy eigenvalues of this Hamiltonian.

c) Draw an energy level diagram for the case $k >> \frac{m_1 m_2}{m_1 + m_2} w^2$.

18. a) Write down the Hamiltonian for a charged particle (e,m) in a constant magnetic field $\vec{B}$. (Hint: minimal coupling)

b) Taking the field direction as the z-axis and adopting any convenient gauge, solve for the eigenvalues of this system.

c) What are the degeneracies of each eigenvalue E?

d) Suppose an additional electric field $\vec{F}$ is applied, parallel to $\vec{B}$. What is the new Hamiltonian and how are the eigenvalues modified?

19. A spin-1/2 particle is in an eigenstate of S_y with eigenvalue $\hbar/2$. A magnetic field **B** is applied along the z-direction for a time T and the particle allowed to precess. At that point, the field is rapidly switched into the x-direction and the particle allowed to precess for another interval T. What is the probability that an S_y measurement now will show that the initial spin has flipped to $-\hbar/2$?

20. All parts of this question call for quick and short answers.

(a) How do the relativistic corrections to energy levels in the iron atom (Z = 26) compare with those in the hydrogen atom?

(b) For a two-electronic configuration $2p3p$, what are the allowed values of total S, L, and J?

(c) In the above, what would it be for the configuration $2p^2$, upon taking the Pauli Principle into account? Express your answer in the state notation ${}^{2S+1}L_J$.

(d) Explain in a couple of lines why only the excited states of the hydrogen atom and no other exhibit the linear Stark Effect, that is, show energy corrections linear in the field strength of an applied electric field.

(e) Which of the following would be reasonable trial functions for a Rayleigh-Ritz variational estimate of the ground state in a one-dimensional potential: $x + a$, e^{-ax}, e^{-ax2}, $\frac{1}{x^2+a^2}$, where a is a variational parameter.

21. Using $\Psi(x) = Ne^{-\lambda x^2}$ as a trial wavefunction, use the variational principle to estimate the energy of the ground state of a one-dimensional harmonic oscillator. Write down the properly normalized wave function for the ground state.

Note: $\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$ $\qquad$ $\int_{-\infty}^{+\infty} x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

22. a) Write down the Pauli spin matrices σ_i.

b) Starting with the commutation relations obeyed by all angular momentum operators, prove that

$$\sigma_i \sigma_j = i\,\varepsilon_{ijk}\sigma_k + \delta_{ij} I .$$

c) Let $\vec{A}$ and $\vec{B}$ be operators that commute with the Pauli spin matrices. Prove that

$$(\vec{\sigma} \bullet \vec{A})(\vec{\sigma} \bullet \vec{B}) = \vec{A} \bullet \vec{B} + i\vec{\sigma} \bullet (\vec{A} \times \vec{B}).$$

23. None of the questions below involves extensive computation, and most may be answered in a few words. In particular questions 1-4 are either true or false. If you should respond to any of these four questions with "true", then no explanation is necessary. If you should respond with "false", then support your answer with a mathematical proof or a counterexample.

(1) All hermitian operators representing physical observables have inverses.

(2) Unitary operators constitute a subset of Hermitian operators; that is all unitary operators are Hermitian, but not all hermitian operators are unitary.

(3) The product of two unitary operators is unitary.

(4) Let $\hat{\Omega}$ and $\hat{\Lambda}$ be commuting hermitian operators. Then all eigenfunctions of $\hat{\Omega}$ are simultaneously eigenfunctions of $\hat{\Lambda}$.

(5) Do the operations of inversion and conjugation commute? That is, assuming that the operator $\hat{\Lambda}$ possesses an inverse, does $\left(\hat{\Lambda}^{-1}\right)^{\dagger} = \left(\hat{\Lambda}^{+}\right)^{-1}$?

24. The ground state wave function of the hydrogen atom, with potential energy

$$V = -\frac{e^2}{r}$$

has the functional form

$$\psi = Ne^{-r/a_0},$$

with the Bohr radius a_0, and N a normalization factor.

a) Using the s wave radial Schrödinger equation, find a_0 and the energy E in terms of the mass m, charge e, and $\hbar$.

b) Normalize the wave function to represent one particle, determining N. Calculate the electric current density.

c) Given that the proton radius is a factor 10^{-5} smaller than the Bohr radius, calculate the probability of finding the electron in the nucleus. Give your result to one significant figure.

25. A spin - $\frac{1}{2}$ charged particle is in the $s_z = -\frac{\hbar}{2}$ state at $t = 0$ in a constant magnetic field $\vec{B}_0 = (0,0,B_0)$. A weak, rotating magnetic field $\vec{B}_1 = (B_1\cos\omega t, B_1\sin\omega t, 0)$ is switched on, with $B_1 << B_0$.

a) Calculate the probability that the spin state is $s_z = \frac{\hbar}{2}$ at time t.

b) When would you expect perturbation theory to break down?

Hint: It will be useful to work with the operators $s_{\pm}$ when evaluating matrix elements.

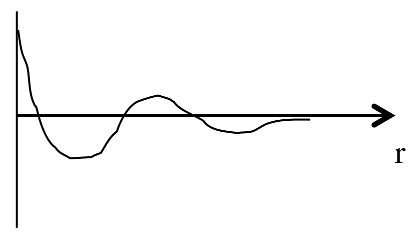
26. a) Evaluate the elastic differential scattering cross-section in the first Born approximation for the scattering of particles of mass m from a Yukawa potential $\frac{A}{r}e^{-\alpha r}$. Express your results in terms of the momentum transfer $\vec{q}$.

b) Also compute the total cross-section.

c) What happens to these results as $\alpha \to 0$?

27. All parts of this question call for quick and short answers.

a) Identify the state of the hydrogen atom whose radial function is



b) Identify which of the operators $\{L^2, L_Z, parity\}$ have sharp eigenvalues in the state described by $\frac{1}{\sqrt{6}}Y_1^1 + \frac{1}{\sqrt{6}}Y_3^1 + \sqrt{\frac{2}{3}}Y_5^1$, where Y_ℓ^m is a standard spherical harmonic.

c) Estimate the radius of the n=100 state of the hydrogen atom.

d) If the natural width of an atomic state is 1μeV, estimate its lifetime.

e) Which of the following operators is Hermitian:

$$i\frac{\partial^3}{\partial x^3},\ x\left(p^2+x^2\right)x,\ \left(x\frac{\partial}{\partial x}-\frac{\partial}{\partial x}x\right),\ \vec{x}\times\vec{p}$$

f) Which of the following are possible variational trial functions for the ground state of a one-dimensional potential well:

$$e^{-ax},\ \frac{1}{x^2+\alpha^2},\ e^{-\alpha x^2}.$$

28. a) A spin-1/2 electron is in a uniform magnetic field $\vec{B}_o = B_o\hat{z}$. At time $t = 0$ the spin is pointing in the x-direction, i.e., $|S_x(t{=}0)\rangle = \frac{\hbar}{2}$. The gyromagnetic ratio is γ, and a reference frequency is defined by $\omega_o \equiv \gamma B_o$. Calculate the expectation value $\langle S_x(t)\rangle$ at time t.

b) An additional magnetic field $\vec{B}_1 = B_1[\cos(\omega_o t)\hat{x} + \sin(\omega_o t)\hat{y}]$ is now applied. If an electron in the combined field $\vec{B}_o + \vec{B}_1$ has spin pointing along $+\hat{z}$ at time t=0, what is the probability that it will have flipped to $-\hat{z}$ at time t?

29. A non-relativistic electron of mass m is confined to move in one dimension. Its wave function $\psi(x)$ obeys the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

a) Consider a delta function barrier at $x = 0$, described by $V(x) = V_o\delta(x)$. Show that the equations which relate the wave function and its first derivative on the left (L) and right (R) sides of the potential barrier have the following form:

$$\psi_R(0) = \psi_L(0) = \psi(0)$$

$$\left[\frac{d\psi_R}{dx}\right]_{x=0} - \left[\frac{d\psi_L}{dx}\right]_{x=0} = A\psi(0)$$

b) Give an expression for A in the last equation of part (a).

c) A beam of electrons of mass m is incident on the delta function potential of part (a). The wave function on the left and right sides is written as

$$\psi_L(x) = e^{ikx} + ae^{-ikx}$$

$$\psi_R(x) = be^{ikx}$$

Which way is the beam traveling? What is the speed of the electrons?

d) Give an expression for the transmission coefficient $T = |b|^2$ as a function of A and k.

30. A spin-1/2 particle is subject to a static magnetic field $\vec{B}$ along the x-direction. This give rise to a Hamiltonian

$$\hat{H} = -\left(\frac{eB}{m}\right)\hat{S}_x .$$

Assume the initial state is $\left|\chi(t=0)\right\rangle = \left|+z\right\rangle$.

a) Calculate $\left|\chi(t)\right\rangle$

b) Calculate $\left\langle \hat{S}_z \right\rangle$ as a function of time. What is the precession frequency?

c) Calculate the time-dependent uncertainty in $\hat{S}_z$, $(\Delta S_z)^2 = \left\langle \hat{S}_z^2 \right\rangle - \left\langle \hat{S}_z \right\rangle^2$.

31. Given

$$\left[J_i, J_j\right] = i\hbar \sum_k \varepsilon_{ijk} J_k ,$$

$$J_z\left|\alpha\right\rangle = \hbar m\left|\alpha\right\rangle,$$

$$J^2\left|\alpha\right\rangle = \hbar^2 j(j+1)\left|\alpha\right\rangle,$$

$$\left\langle \alpha | \alpha \right\rangle = 1$$

a) Calculate $\left\langle \alpha \middle| J_i \middle| \alpha \right\rangle$

b) Calculate $\Delta J_i \equiv \sqrt{\left\langle \alpha \middle| J_i^2 \middle| \alpha \right\rangle - \left\langle \alpha \middle| J_i \middle| \alpha \right\rangle^2}$.

c) If measurements of L^2 and L_z are made on a state whose wave function is $Ae^{-br}\sin\theta\cos\varphi$, what are the possible values found?

32. A lithium atom in its ground s state is placed in a small static electric field ε. By spectroscopic measurement, the ground state energy is found to shift in energy according to

$$\Delta E = -\frac{1}{2}\alpha\varepsilon^2 ,$$

where α is a constant, the atomic polarizability. From the experiment it is found that $\alpha = 24$ (Å)^3, where Å is an Angstrom.

a) Assuming an electric dipole interaction $H' = -\hat{\mu}\varepsilon$, use the following information to derive a simple formula for α:

- The nearest excited state to the ground state is a p state with an energy separation of $\hbar\omega$.

-The only relevant non-vanishing dipole matrix element is $\langle p|\hat{\mu}|s\rangle = \mu_0$.

b) The spontaneous lifetime of the p state, $\tau = 27ns$, is related to the dipole transition moment by the Einstein A coefficient,

$$A = \frac{1}{\tau} = \frac{4}{3}\frac{\mu_0^2 k^3}{\hbar}$$

where $k = 2\pi/\lambda$ and $\lambda = 670nm$. Use this information to write a formula for α in terms of λ, τ, and the speed of light c. Evaluate the result, either with a calculator or by estimate, to show that it agrees almost exactly with the measured value of α.

33. All parts of this question call for quick and short answers.

a) Give to within a factor of 2 the radius of the n = 100 Bohr orbit in the hydrogen atom.

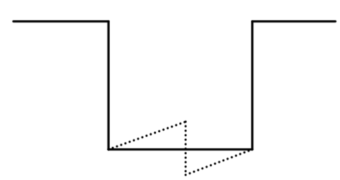
b) Give to within a factor of 2 the binding energy of the ground state when a proton and anti-proton are bound by their Coulomb interaction.

c) Identify which of the operators $\{L^2, L_z, \text{parity}\}$ have definite values in the state described by $\frac{1}{\sqrt{6}}Y_{11} + \frac{1}{\sqrt{6}}Y_{31} + \sqrt{\frac{2}{3}}Y_{51}$, where the $Y_{\ell m}$ are standard spherical harmonics.

d) If the natural width of an atomic state is 1μeV, estimate its lifetime.

e) Which of the following operators is Hermitian:

$$i\frac{\partial^3}{\partial x^3},\ x(p^2+x^2)x,\ \vec{r}\times\vec{p}.$$



f) A one-dimensional potential well is perturbed as indicated by the dashed line. What order and sign of the perturbation correction do you expect for the ground state energy?

34. a) An electron $\left(\text{spin} = \frac{1}{2}\right)$ is prepared in an eigenstate of S_x with eigenvalue $+\hbar/2$ and subjected to a uniform magnetic field $\vec{B} = (0, 0, B)$ for a time T. At that point, the field is suddenly rotated through 90° to $\vec{B} = (0, B, 0)$. After another time interval T, the electron's spin in the x-direction is measured. What is the probability of obtaining the value $-\hbar/2$? Hint: It will be useful to consider S_z eigenstates for the first time interval and S_y eigenstates for the second.

b) For B = 200 G, at what earliest value of T will this probability be a maximum?

35. At time $t = 0$, a hydrogen atom is in the state

$$|t = 0\rangle = \frac{1}{\sqrt{2}}|1s0\rangle - \frac{i}{3\sqrt{2}}|2p1\rangle + \frac{1}{3\sqrt{2}}|2p-1\rangle + \frac{\sqrt{7}}{3\sqrt{2}}|2p0\rangle,$$

where the kets represent normalized $|n\ell m\rangle$ states of the atom.

a) What values of angular momentum L^2 will be found upon measurement in this state?

b) What is the expectation value $\langle L^2\rangle$ in this state?

c) If no measurements are made, what is the state $|t\rangle$ at a later time t?

d) If a measurement of L_z at $t = 0$ yields $\hbar$, what is the subsequent time evolution of the state?

e) If a weak electric field were applied at $t = 0$, will the state exhibit a linear Stark effect? Explain.

36. All parts of this question call for quick and short answers.

a) What kind of electromagnetic transition occurs between the $4f_0$ and $1s_0$ states (notation: $n\ell_m$) of the hydrogen atom?

b) The 589 nm yellow line of sodium arises from an excited state with a lifetime for optical emission of 10^{-8} s. Estimate the natural width of the line.

c) If you are told that in a certain reaction, the electron comes out with its spin always parallel to its momentum, argue that parity conservation is violated.

d) Consider three identical particles in a system which has only three states a,b and c. How many distinct allowed configurations are there if the particles are (i) bosons, (ii) fermions?

e) Electron capture by the nucleus involves the absorption by the nucleus (Z, A) of one of the atomic electrons. Which electrons (s, p, d, f, etc) would you expect to be dominantly involved? How does the capture probability scale with Z?

37. Consider a particle of mass m in a potential well

$$\begin{aligned} V(x) &= 0 \qquad & -a \leq x \leq a \\ &= \infty \qquad & |x| > a \end{aligned}$$

a) Using the simplest even polynomial that vanishes at $x = \pm a$, namely

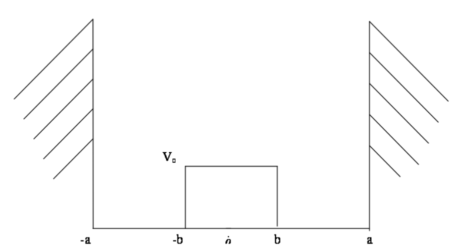
$$\begin{aligned} \psi_t &= N(a^2 - x^2) \qquad & -a \leq x \leq a \\ &= 0 \qquad & |x| > a \end{aligned} \quad ,$$

where N is a constant, calculate variationally the ground state energy of the particle.

b) What is the exact energy for the ground state? Compare with the estimate in (a).

c) To get the first excited state in the well, what is the simplest polynomial which you would use as a trial function?

38. Give a particle of mass m in a one-dimensional square well potential with a small perturbing potential as shown in the drawing,



a) Obtain the normalized unperturbed wave functions and their energies.

b) Find the first order correction to the energy of the ground state.

c) Find the first order correction to the wave function of the ground state.

39. The energy levels of the hydrogen atom from the Schrödinger equation with Coulomb potential are highly degenerate. They depend only on the radial quantum number n, but are independent of the other quantum numbers such as the orbital angular momentum L, the spin S and the total angular momentum J. However, the observed levels show small splittings between the degenerate levels.

These splittings are known as fine structure, hyperfine structure and the Lamb shift.

a) Explain the physical origin of these splittings. What interactions in addition to the Coulomb interation if any are responsible for these splittings?

b) Why are these splittings small? Order these splittings in terms of the order of magnitude of their contributions.

c) Draw an energy level diagram displaying these splittings. Label each energy level with the appropriate quantum numbers, such as n, L, S, J. the ordering of the levels should be taken into account.

40. A system is in an eigenstate of the operator J^2 with eigenvalue $2\hbar$.

a) By measuring J_z additionally the pure state $|j,m_z\rangle$ is prepared. Discuss without computing explicit numbers the possible values $\hbar m_x$ of a measurement of J_x that is performed after J_z has been measured.

b) What is the probability of the values discussed in (a).

Hint: Use the $j=1$ matrix representation for J_x given by

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

c) After the measurement of J_x the angular momentum component J_z is measured again. What is the probability of measuring the former values $\hbar m_z$ again?

41. The Hamiltonian of the harmonic oscillator can be written in the following form

$$H = H_0 + H_1;\ H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega x^2,\ \ H_1 = \alpha\frac{1}{2}m\omega x^2,$$

where p and x denote the corresponding momentum and position operators. The result obtained from perturbation theory can be easily compared to the exact result of the harmonic oscillator.

a) Compute the energy correction $E_n^{(1)}$ for the system in first order perturbation theory.

b) What is the value of the energy correction $E_n^{(2)}$ in second order perturbation theory?

c) Compare the energy corrections obtained in (a) and (b) with the exact energy eigenvalues of the harmonic oscillator associated with the Hamiltonian H.

42. Consider a one-dimensional square well potential of width a and depth V_0. We intend to study the properties of the bound state of a particle in this well when its width a approaches zero.

a) Show that there indeed exists only one bound state and calculate its energy E. Check that E varies with the square of the area aV_0 of the well.

b) How can the preceding considerations be applied to a particle placed in the potential $V(x) = -a\delta(x)$.

43. Consider an electron of a linear triatomic molecule formed by three equidistant atoms. We use $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ to denote three orthonormal states of this electron, corresponding respectively to three wave functions localized about the nuclei of atoms *A, B, C*. We shall confine ourselves to the subspace of the state space spanned by $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$.

When we neglect the possibility of the electron jumping from one nucleus to another, its energy is described by the Hamiltonian H_0 whose eigenstates are the three states $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ with the same eigenvalue E_0. The coupling between the states $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ is described by an additional Hamiltonian W defined by:

$$
\begin{aligned}
W|\varphi_A\rangle &= -a|\varphi_B\rangle, \\
W|\varphi_B\rangle &= -a|\varphi_A\rangle - a|\varphi_C\rangle, \\
W|\varphi_C\rangle &= -a|\varphi_B\rangle,
\end{aligned}
$$

where a is a real positive constant.

a) Calculate the energies and eigenstates of the Hamiltonian $H = H_0 + W$.

b) Let D be the observable whose eigenstates are $|\varphi_A\rangle, |\varphi_B\rangle, |\varphi_C\rangle$ with respective eigenvalues $-d, 0, d$. Do the operators D and H commute?

44. The transition $^2P_{3/2} \rightarrow {}^2S_{1/2}$ is observed in a magnetic field of 1 Tesla.

(a) Sketch the Zeeman splitting of both states.
(b) What are the Lande g factors of the two states?
(c) Estimate the magnitude of the splitting between the levels.

45. An electron is in the spin state $N\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$.

(a) Determine the normalization constant N.
(b) If you measured S_z on this state, what values would you get and with what probabilities? What is the expectation value?
(c) If you measured S_x on this state, what values would you get and with what probabilities? What is the expectation value?
(d) If you measured S_y on this state, what values would you get and with what probabilities? What is the expectation value?

46. All parts of this question call for quick and short answers.

(a) What is the wavelength of the Ly_α transition in the hydrogen atom, from $n = 2$ to $n = 1$?

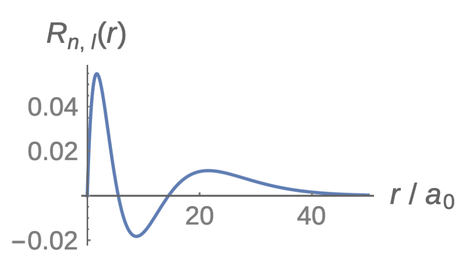
(b) Identify the parities of the following:

$$x^3 + 3x^2y + z,\ re^{-r}\sin\theta\cos\phi,\ \tfrac{1}{2}Y_0^0 + \tfrac{\sqrt{3}}{2}Y_1^0,\ |2p, m=0\rangle$$

of the hydrogen atom.

(c) Find the eigenvalues of (can be done almost by inspection) $\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(d) Evaluate $\sigma_x\sigma_y\sigma_z$, these being the usual Pauli spin matrices.
(e) For the radial wavefunction of the hydrogen atom shown, what are the values of n and l?



47.

(a) Consider an angular momentum –1 system described in the l_z-representation by

$$(1/\sqrt{26})\begin{pmatrix}1\\4\\-3\end{pmatrix}$$

What is the probability that an l_x measurement gives 0?

(b) Using any convenient representation, find the eigenvalues and eigenvectors of the $l_x l_y + l_y l_x$.

48. A hydrogen atom is prepared at time 0 in the state

$$[2|100\rangle + |211\rangle - |200\rangle + 2|320\rangle]/\sqrt{10},$$

in the usual $|nlm>$ notation.

(a) What values of energy would be seen upon an energy measurement and with what probabilities?

(b) What is the expectation value of the energy?

(c) For an l^2 measurement, what values and with what probabilities would you expect?

(d) If no measurement is made, what is the state at a later time t?

49. An electron is at rest in an oscillating magnetic field $\vec{B} = B_0 \cos\omega t\ \hat{k}$. . Construct the Hamiltonian for this system. Suppose the electron starts at $t = 0$ in the up-state wrt x, that is,

$$(1/\sqrt{2})\begin{pmatrix}1\\1\end{pmatrix}.$$

Solve the time-dependent Schrödinger equation to get the probability of finding the electron in the orthogonal "down" state at a later time t.

49. Instead of the z-representation of Pauli matrices with σ_z diagonal, suppose you transformed to the x-representation by diagonalizing σ_x. Construct explicitly the similarity transformation that accomplishes this, and then examine what happens to the other two Pauli matrices in this new representation. Examine your result to see its geometrical interpretation in terms of original and transformed axes.

50. (a) Suppose you wanted to describe an unstable particle that spontaneously decays with a "lifetime" τ . In this case, the total probability of finding the particle somewhere should not be a constant but decrease, say exponentially, in time: $P(t) = \int |\psi(x,t)|^2 dx = e^{-t/\tau}$. A way of reaching this is to assume that the potential V in the Schrödinger equation is not real but complex, $V = V_0 - i\,\Gamma$, with

V_0 and Γ positive real constants. Show that the equation of continuity now gives $\frac{dP}{dt} = -\left(\frac{2\Gamma}{\hbar}\right) P$, and relate the lifetime τ to the "linewidth" Γ.

(b) What is the energy uncertainty or linewidth in eV of a particle that has a decay lifetime of 12 mins. What particle is this?

51. For a particle in a box (0, a), consider an initial normalized wave function $\psi(x, 0) = N\,[\sin(\pi x/a)]^5$.

(a) Determine N.
(b) Find $\psi(x, t)$ at a later time t.
(c) What is the probability that an energy measurement on the state gives E_3?

Hint: There is no need to integrate $\sin^{10}(\theta)$ if you write sin in exponentials in θ, and further powers in a power series expansion.

52. (a) When two different states share the same energy, they are said to be "degenerate." Prove that one-dimensional bound states of a quantum system in an arbitrary potential $V(x)$ do not have such degeneracy.

Hint: Assume this were not true, write the time-independent Schrödinger equation for both ψ_1 and ψ_2, multiply each by the other function, subtract, and integrate over all x. Keep in mind that bound state wave functions vanish at $\pm\infty$ to be normalizable, and follow the logic to conclude that ψ_1 and ψ_2 are not distinct.

(b) Establish the result that between any two successive nodes of a one-dimensional bound state wave function $\psi_m(x)$, the wave function $\psi_n(x)$ of a higher energy state, $E_n > E_m$, will have at least one node.

Hint: As in the previous problem, write the Schrödinger equations for the two states, multiply one by the other function and subtract to get $\psi_m'(x_2)\psi_n(x_2)$-$\psi_m'(x_1)\psi_n(x_1) = 2m/\hbar^2(E_n - E_m)\int_{x_1}^{x_2}\psi_m\,\psi_n\,dx$.

53. The Hamiltonian of a three-level system is given by

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Two other observables are represented by the matrices

$$A = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω, λ, and μ are real, positive numbers.

(a) Which of A or B would you use to get a unique classification of the eigenstates of the system, and why?

(b) Find that unique labeling, giving both Dirac ket labels for the states and their column vector representations.

54. A one-dimensional harmonic oscillator with

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

is perturbed by a time-independent potential $-\lambda x^4$, where λ is a small positive constant.

(a) Calculate the leading order correction to the energy of the *n*-th state.

(b) For what values of λ would you expect your answer to be accurate?

(c) Were you to proceed to the next order correction, which intermediate states would contribute? You do not need to carry out the calculations.